A new modified Weibull distribution

Saad J. Almalki *, Jingsong Yuan

School of Mathematics, University of Manchester, Manchester M13 9PL, UK

A R T I C L E   I N F O

Article history:
Received 2 February 2012
Received in revised form 29 October 2012
Accepted 30 October 2012
Available online 22 November 2012

Keywords:
Weibull distribution
Additive Weibull
Modified Weibull
Maximum likelihood estimation

A B S T R A C T

We introduce a new lifetime distribution by considering a serial system with one component following a Weibull distribution and another following a modified Weibull distribution. We study its mathematical properties including moments and order statistics. The estimation of parameters by maximum likelihood is discussed. We demonstrate that the proposed distribution fits two well-known data sets better than other modified Weibull distributions including the latest beta modified Weibull distribution. The model can be simplified by fixing one of the parameters and it still provides a better fit than existing models.

© 2012 Elsevier Ltd. All rights reserved.

1. Introduction

The Weibull distribution [28] has been used in many different fields with many applications, see for example [18]. The hazard function of the Weibull distribution can only be increasing, decreasing or constant. Thus it cannot be used to model lifetime data with a bathtub shaped hazard function, such as human mortality and machine life cycles. For many years, researchers have been developing various extensions and modified forms of the Weibull distribution, with number of parameters ranging from 2 to 5. The two-parameter flexible Weibull extension of Bebbington et al. [5] has a hazard function that can be increasing, decreasing or bathtub shaped. Zhang and Xie [31] studied the characteristics and application of the truncated Weibull distribution which has a bathtub shaped hazard function. A three-parameter model, called exponentiated Weibull distribution, was introduced by Mudholkar and Srivastave [17]. Another three-parameter model is by Marshall and Olkin [15] and called extended Weibull distribution. Xie et al. [30] proposed a three-parameter modified Weibull extension with a bathtub shaped hazard function. The modified Weibull (MW) distribution of Lai et al. [13] multiplies the Weibull cumulative hazard function by $e^{\gamma x}$, which was later generalized to exponentiated form by Carrasco et al. [6]

$$F(x) = (1-e^{-x^a})^\beta, \quad x \geq 0.$$  

(1)

Recent studies of the modified Weibull include [11,26,27].

Among the four-parameter distributions, the additive Weibull distribution (AddW) of Xie and Lai [29] with cumulative distribution function (CDF)

$$F(x) = 1-e^{-x^{a}}e^{\gamma x}, \quad x \geq 0,$$

has a bathtub-shaped hazard function consisting of two Weibull hazards, one increasing ($0>1$) and one decreasing ($0<\gamma<1$). The modified Weibull distribution of Sarhan and Zaindin (SZMW) [21] can be derived from the additive Weibull distribution by setting $\theta = 1$. A four-parameter beta Weibull distribution was proposed by Famoye et al. [10]. Cordeiro et al. [8] introduced another four-parameter called the Kumaraswamy Weibull distribution.

Five-parameter modified Weibull distributions include Phani’s modified Weibull [20], the beta modified Weibull (BMW) introduced by Silva et al. [24] and further studied by Nadarajah et al. [19]. The latest examples include the beta generalized Weibull distribution by Singla et al. [25], exponentiated generalized linear exponential distribution by Sarhan et al. [22] and the generalized Gompertz distribution by El-Gohary et al. [9].

We propose a new lifetime distribution based on the Weibull and the modified Weibull (MW) distributions by combining them in a serial system. The hazard function of the new distribution is the sum of a Weibull hazard function and a modified Weibull hazard function. Section 2 gives definition, motivation and usefulness of this model and lists its sub-models. Section 3 considers properties of the new distribution such as hazard, moments and order statistics. Section 4 discusses estimation of the parameters. Two real data sets are analyzed in Section 5 and the results are compared with existing distributions. Section 6 concludes the paper.

* Corresponding author. Tel.: +44 7598894919.
E-mail address: saad.al-malki-2@postgrad.manchester.ac.uk (S.J. Almalki).

0951-8320/$ - see front matter © 2012 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ress.2012.10.018
2. The model

2.1. Definition

We define a new modified Weibull distribution (NMW) by the following CDF:

\[ F(x) = 1 - e^{-x^\alpha - \beta x^\gamma e^{\delta x}}, \quad x \geq 0, \]  

where \( \alpha, \beta, \gamma, \) and \( \delta \) are non-negative, with \( \theta \) and \( \gamma \) being shape parameters and \( \alpha \) and \( \beta \) being scale parameters and \( \lambda \) acceleration parameter.

The probability density function (PDF) is

\[ f(x) = (\alpha \beta x^{\alpha-1} + \beta (\gamma + 2 \chi \gamma^{-1} e^{x}) e^{-x^\alpha - \beta x^\gamma e^{\delta x}}), \quad x > 0. \]  

It can be rewritten as

\[ f(x) = [h_W(x; \alpha, \beta) + h_{MW}(x; \gamma, \delta)] S_{MW}(x; \alpha, \beta, \gamma, \delta), \]  

where \( S_W, h_W, S_{MW} \) and \( h_{MW} \) are survival and hazard functions of the Weibull and modified Weibull distributions, respectively.

This function can exhibit different behavior depending on the values of the parameters when chosen to be positive, as shown in Fig. 1.

2.2. Sub models

This distribution includes sub models that are widely used in survival analysis. Table 1 shows a list of models that can be derived from the NMW distribution.

![Fig. 1. Probability density functions of the NMW.](image)

Table 1

<table>
<thead>
<tr>
<th>Model</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \theta )</th>
<th>( \lambda )</th>
<th>( S(x) )</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additive Weibull</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( e^{-x^\lambda} )</td>
<td>Xie and Lai [29]</td>
</tr>
<tr>
<td>Modified Weibull</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( e^{-x^\lambda} )</td>
<td>Lai et al. [13]</td>
</tr>
<tr>
<td>S-Z modified Weibull</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( e^{-x^\lambda} )</td>
<td>Sarhan and Zaindin [21]</td>
</tr>
<tr>
<td>Linear failure rate</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( e^{-x^\lambda} )</td>
<td>Bain [4]</td>
</tr>
<tr>
<td>Extreme-value</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( e^{-x^\lambda} )</td>
<td>Bain [4]</td>
</tr>
<tr>
<td>Weibull</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( e^{-x^\lambda} )</td>
<td>Weibull [28]</td>
</tr>
<tr>
<td>Rayleigh</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( e^{-x^\lambda} )</td>
<td>Bain [4]</td>
</tr>
<tr>
<td>Exponential</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( e^{-x^\lambda} )</td>
<td>Bain [4]</td>
</tr>
</tbody>
</table>

![Fig. 2. Hazard functions of the NMW.](image)

2.3. Motivation and interpretation

The survival function of the new distribution is given by

\[ S(x) = e^{-x^\alpha - \beta x^\gamma e^{\delta x}}, \quad x \geq 0, \]  

and, the hazard function is

\[ h(x) = \alpha \beta x^{\alpha-1} + \beta (\gamma + 2 \chi \gamma^{-1} e^{x}) e^{-x^\alpha - \beta x^\gamma e^{\delta x}}, \quad x > 0, \]  

which can be interpreted as that of a serial system with two independent components, one of which follows the Weibull distribution with parameters \( \alpha \) and \( \theta \), and the other follows the modified Weibull distribution of Lai et al. [13] with parameters \( \beta, \gamma, \) and \( \lambda \). Therefore the distribution can be used when there are two types of failure, e.g. a ‘normal’ type and a premature type.

The purpose of the Weibull component is to provide a decreasing hazard function when required, as in the additive Weibull [29], by choosing \( \theta < 1 \). (It will be increasing when \( \theta > 1 \).)

The modified Weibull component has either an increasing or a bathtub shaped hazard function. Together they provide a bathtub shaped hazard function. Together they provide a bathtub shaped hazard function (unless both hazards are increasing) with more flexibility than the additive Weibull. The flexibility is useful when there is a second peak in the distribution as shown in Section 5.

3. Properties of the model

3.1. The hazard function

The hazard function can have many different shapes, including bathtub, as shown in Fig. 2. We can deduce from (6) that it is increasing if \( \theta, \gamma \geq 1 \), decreasing if \( \theta, \gamma < 1 \) and \( \lambda = 0 \) and bathtub shaped otherwise.

It is desirable for a bathtub shaped hazard function to have a long useful life period [12], with relatively constant failure rate in the middle. A few distributions have this property, so does the NMW as shown in Fig. 3.

3.2. The moments

It is customary to derive the moments when a new distribution is proposed. Using the Taylor expansion of \( e^x \) twice, the \( r \)th non-cental moment of the NMW is

\[ \mu_r = \int_0^\infty x^r \, dF(x) \]

\[ = \int_0^\infty x^r \, e^{-x^\alpha - \beta x^\gamma e^{\delta x}} \, dx \]
\[ f(x) = (1 - e^{-\theta x}) = e^{-\theta x} \]

\[ f(x) = \frac{1}{B(r, n-r+1)} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (-1)^{n-\ell} h(x)e^{-(n+\ell+1-\ell)\theta x} \]

\[ f(x) = 2x^\alpha + \beta x e^{\lambda x} \]

\[ f(x) = e^{-\theta x} \]

\[ f(x) = 2x^\alpha + \beta x e^{\lambda x} \]

**Fig. 3.** Hazard functions of the NMW with long useful life period.

**Table 2**

<table>
<thead>
<tr>
<th>Model</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( \lambda )</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMW</td>
<td>0.071</td>
<td>7.015 \times 10^{-8}</td>
<td>0.016</td>
<td>0.595</td>
<td>0.197</td>
</tr>
<tr>
<td>(( \theta = 0 ))</td>
<td>(0.031)</td>
<td>(1.501 \times 10^{-7})</td>
<td>(3.002)</td>
<td>(0.128)</td>
<td>(0.184)</td>
</tr>
<tr>
<td>MW</td>
<td>0.062</td>
<td>0.027</td>
<td>0.356</td>
<td>0.013</td>
<td>0.023</td>
</tr>
<tr>
<td>(( \alpha = 0 ))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AddW</td>
<td>1.138 \times 10^{-8}</td>
<td>0.086</td>
<td>0.477</td>
<td>4.214</td>
<td></td>
</tr>
<tr>
<td>(( \delta = 0 ))</td>
<td>(0.036)</td>
<td></td>
<td>(1.012)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SZMW</td>
<td>0.013</td>
<td>8.408 \times 10^{-9}</td>
<td>4.224</td>
<td>0.023</td>
<td></td>
</tr>
<tr>
<td>(( \beta = 0 ))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The maximum likelihood estimates can be obtained by solving the non-linear equations numerically for $x, \beta, \theta, \gamma$ and $\lambda$. This can be done using R, Matlab and Mathcad, among other packages. The relatively large number of parameters can cause problems especially when the sample size is not large. A good set of initial values is essential.

We have also obtained all the second partial derivatives of the log-likelihood function for the construction of the Fisher information matrix, so that standard errors of the parameter estimates can be obtained in the usual way. These are in the Appendix.

### 5. Applications

In this section we provide results of fitting the NMW to two well-known data sets and compare its goodness-of-fit with other modified Weibull distributions using Kolmogorov–Smirnov (K–S) statistic, as well as Akaike information criterion (AIC) [2] and Bayesian information criterion (BIC) [23] values.

**Table 3**

Log-likelihood, K–S statistic, the corresponding P-values, AIC and BIC values of models fitted to Aarst data for comparison with beta modified Weibull (Silva et al. [24]).

<table>
<thead>
<tr>
<th>Model</th>
<th>Log-lik</th>
<th>K–S</th>
<th>P-value</th>
<th>AIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMW</td>
<td>–212.90</td>
<td>0.088</td>
<td>0.803</td>
<td>435.8</td>
<td>445.4</td>
</tr>
<tr>
<td>MW</td>
<td>–227.16</td>
<td>0.129</td>
<td>0.346</td>
<td>460.3</td>
<td>466.0</td>
</tr>
<tr>
<td>AddW</td>
<td>–221.51</td>
<td>0.127</td>
<td>0.365</td>
<td>451.0</td>
<td>458.7</td>
</tr>
<tr>
<td>SZMW</td>
<td>–229.88</td>
<td>0.151</td>
<td>0.365</td>
<td>465.8</td>
<td>471.5</td>
</tr>
<tr>
<td>BMW</td>
<td>–220.80</td>
<td>0.127</td>
<td>0.365</td>
<td>451.6</td>
<td>461.2</td>
</tr>
</tbody>
</table>

**Table 4**

MLEs of parameters and corresponding standard errors in brackets for the Meeker and Escobar data.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\hat{a}$</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\gamma}$</th>
<th>$\hat{\theta}$</th>
<th>$\hat{\lambda}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMW</td>
<td>0.024</td>
<td>$5.991 \times 10^{-8}$</td>
<td>0.012</td>
<td>0.629</td>
<td>0.056</td>
</tr>
<tr>
<td>MW</td>
<td>0.018</td>
<td>$8.164 \times 10^{-8}$</td>
<td>0.018</td>
<td>0.454</td>
<td>0.024</td>
</tr>
<tr>
<td>(a = 0, $\theta = 0$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AddW</td>
<td>$1.320 \times 10^{-7}$</td>
<td>0.019</td>
<td>0.604</td>
<td>2.830</td>
<td>7.133 $\times 10^{-3}$</td>
</tr>
<tr>
<td>(l = 0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SZMW</td>
<td>$2.939 \times 10^{-3}$</td>
<td>$1.497 \times 10^{-9}$</td>
<td>3.585</td>
<td>3.134</td>
<td></td>
</tr>
<tr>
<td>(\theta = 1, \lambda = 0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 4. For Aarst data: (a) hazard function, (b) TTT-transform plot, (c) pdf and (d) survival function using NMW plus sub models, and beta modified Weibull.
5.1. Aarset data

The data represent the lifetimes of 50 devices [1]. Many authors have analysed this data set, including Mudholkar and Srivastava [17], Xie and Lai [29], Lai et al. [13], Sarhan and Zaindin [21], and Silva et al. [24]. It is known to have a bathtub-shaped hazard function (Fig. 3a) as indicated by the scaled TTT-Transform plot (Fig. 3b). Table 2 gives ML estimates of parameters of the NMW and sub-models with standard errors in brackets and goodness of fit statistics are in Table 3. We find that the NMW distribution with the same number of parameters provides a better fit than the beta modified Weibull distribution (BMW) which was the best in Silva et al. [24]. It has the largest likelihood, and the smallest K–S, AIC and BIC values among those considered in this paper. It is clear in Fig. 3c that the NMW fits the left and right peaks in the histogram better and its survival function follows the Kaplan–Meier estimate more closely (Fig. 3d).

Table 5
Log-likelihood, K–S statistic, the corresponding P-values, AIC and BIC values of models fitted to Meeker and Escobar data.

<table>
<thead>
<tr>
<th>Model</th>
<th>Log-lik</th>
<th>K–S</th>
<th>P-value</th>
<th>AIC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>NMW</td>
<td>−166.18</td>
<td>0.148</td>
<td>0.482</td>
<td>344.4</td>
<td>351.4</td>
</tr>
<tr>
<td>MW</td>
<td>−178.06</td>
<td>0.182</td>
<td>0.242</td>
<td>362.1</td>
<td>366.3</td>
</tr>
<tr>
<td>AddW</td>
<td>−178.11</td>
<td>0.191</td>
<td>0.197</td>
<td>364.2</td>
<td>369.8</td>
</tr>
<tr>
<td>SZMW</td>
<td>−177.90</td>
<td>0.186</td>
<td>0.221</td>
<td>361.8</td>
<td>366.0</td>
</tr>
<tr>
<td>BMW</td>
<td>−167.55</td>
<td>0.161</td>
<td>0.378</td>
<td>345.1</td>
<td>352.1</td>
</tr>
</tbody>
</table>

5.2. Meeker and Escobar data

The data are failure and running times of a sample of 30 devices (Meeker and Escobar [16, p. 383]). Two types of failures were observed for this data. It was shown by Nadarajah et al. [19] to be best fit by the beta modified Weibull distribution. The data have a bathtub shaped hazard function (Fig. 4a and b). Again the NMW distribution (Table 4) provides a better fit than the BMW, as can be seen from Table 5 (Fig. 5).

6. Sub-model of the NMW with $\gamma = 1$

To simplify the statistical inference, it is always a good idea to reduce the number of parameters of any distribution and investigate how that affects the ability of the reduced model to fit the data. In this section we reduce the number of parameters from five to four, by setting $\gamma = 1$. We test the reduced model $H_0: \gamma = 1$ against the original model $H_a: \gamma \neq 1$. For each data set, Table 6 shows ML estimates of the four parameter NMW, the log-likelihood value under $H_0$, likelihood ratio statistic (LRT) with $P$-value in brackets, AIC, K–S statistic with $P$-value in brackets.

The likelihood ratio statistics against the full model with five parameters are 1.31 ($P$-value=0.252) and 2.45 ($P$-value=0.118), respectively, on 1 d.f. Therefore we can choose the reduced model with four parameters. The likelihood and AIC value also points to this model when the modified beta distribution is included in the comparison. Fig. 6 shows the reduced model is nearly as good as the full model for both data sets.

Fig. 5. For Meeker and Escobar data: (a) hazard function, (b) TTT-transform plot, (c) pdf and (d) survival function using NMW plus sub models, and beta modified Weibull.
7. Conclusions

A new distribution, based on Weibull and modified Weibull distributions, has been proposed and its properties studied. The idea is to combine two components in a serial system, so that the hazard function is either increasing or more importantly, bathtub shaped. Using a modified Weibull component, the distribution has flexibility to model the second peak in a distribution. We have shown that the new modified Weibull distribution fits certain well-known data sets better than existing modifications of the Weibull distribution. Reducing the number of parameters to four by fixing one of the parameters still provides a better fit than existing models.

Future work includes MCMC methods with censored data, regression problems with covariates and parameter reduction.

Acknowledgments

We would like to thank the referees for their comments and suggestions which improved the presentation of the paper. The first author wishes to thank the Saudi Arabia Culture Bureau in the UK and the Taif University for their financial support.

Appendix A

The log-likelihood function of the NMW(α,β,γ,λ) can be written as

\[ \mathcal{L}(\hat{\beta}) = \sum_{i=1}^{n} \left[ \ln(h(x_i; \hat{\beta})) - 2x_i - \beta x_i^2 e^{x_i} \right] \]

where \( h(x_i; \hat{\beta}) \) is the hazard rate function (6) of the NMW and \( \hat{\beta} = (\alpha, \beta, \gamma, \lambda) \) is the vector of parameters.

The second partial derivatives are as follows:

\[ \mathcal{L}_{\hat{\beta}\hat{\beta}} = -\sum_{i=1}^{n} \left( \frac{h(x_i; \hat{\beta})^2}{h(x_i; \hat{\beta})} \right)^2 \]

Table 6

Results of fitting NMW with \( \gamma = 1 \) to both data sets.

<table>
<thead>
<tr>
<th>Data</th>
<th>( \hat{\alpha} )</th>
<th>( \hat{\beta} )</th>
<th>( \hat{\gamma} )</th>
<th>( \hat{\lambda} )</th>
<th>Log-lik</th>
<th>AIC</th>
<th>LRT (P-value)</th>
<th>K-S (P-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aarset</td>
<td>0.092</td>
<td>2.2 × 10^{-4}</td>
<td>0.531</td>
<td>0.160</td>
<td>-213.56</td>
<td>435.1</td>
<td>1.31</td>
<td>0.105</td>
</tr>
<tr>
<td></td>
<td>(0.039)</td>
<td>(2.1 × 10^{-3})</td>
<td>(0.104)</td>
<td>(0.011)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Meeker</td>
<td>0.017</td>
<td>3.5 × 10^{-4}</td>
<td>0.675</td>
<td>0.039</td>
<td>-167.40</td>
<td>342.8</td>
<td>2.45</td>
<td>0.153</td>
</tr>
<tr>
<td></td>
<td>(0.013)</td>
<td>(4.1 × 10^{-3})</td>
<td>(0.141)</td>
<td>(4.0 × 10^{-3})</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 6. (a and b) Fitted pdf and survival functions for Aarst data and (c and d) those for Meeker and Escobar data, five parameters (solid lines) vs four parameters (dotted lines).
\( \ell_{\beta_{ij}} = \frac{1}{h(x_i; \gamma)} \left( \frac{\partial \ell_{\beta_{ij}}}{\partial \beta_{ij}} \right) \)

\( \ell_{\alpha_{ij}} = \frac{1}{h(x_i; \gamma)} \left( \frac{\partial \ell_{\alpha_{ij}}}{\partial \alpha_{ij}} \right) \)

\( \ell_{\gamma_{ij}} = \frac{1}{h(x_i; \gamma)} \left( \frac{\partial \ell_{\gamma_{ij}}}{\partial \gamma_{ij}} \right) \)

\( \ell_{\Delta_{ij}} = \frac{1}{h(x_i; \gamma)} \left( \frac{\partial \ell_{\Delta_{ij}}}{\partial \Delta_{ij}} \right) \)

\( h_i(x_i; \gamma) = \beta h_{\Delta_{ij}}(x_i; \gamma) \)

\( h_i(x_i; \gamma) = \beta h_{\alpha_{ij}}(x_i; \gamma) \)

\( h_i(x_i; \gamma) = \beta h_{\gamma_{ij}}(x_i; \gamma) \)

\( h_i(x_i; \gamma) = \beta h_{\delta_{ij}}(x_i; \gamma) \)

References